Finite dimensional modules

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In this seminiar, we restrict ourselves that g is a semisimple Lie algebra over $\mathbb{C}$, with a fixed triangular decomposition $g = N^- \oplus H \oplus N^+$. ϕ the corresponding root system. $\Delta = \{\alpha_1 \cdots \alpha_l\}$ a basis of ϕ. $\mathcal{W}$ the Weyl group.

Let V is a finite dimensional g-module. V is a weight module since H acts diagonally on V. Due to Weyl's complete reducibility theorem, every finite dimensional g-module is a direct sum of irreducible ones, it suffices to study finite dimensional irreducible g-modules

Recall:

(up to isomorphism)

Thm: Let $\lambda \in H^*$, there exists only one irreducible standard cyclic module $V(\lambda) = U(g).v_\lambda$ of weight λ.

Since V has at least one maximal vector v_λ, the $V = U(g).v_\lambda$ by irreducibility. That means V is isomorphic to $V(\lambda)$

$\boxed{\text{Q: Which of the } V(\lambda) \text{ are finite dimensional ?}}$

For each simple root α_i, let S_i be the corresponding copy of $sl(2,\mathbb{C})$ in g. See $V(\lambda)$ as a finite dimensional module for S_i. v_λ is also a maximal vector for S_i. For each i, the weight of a maximal vector $\lambda(h_i)$ is a nonnegative integer (one less than $\dim V$), then we have

Thm: If V is a finite dimensional irreducible g-module of highest weight λ, then $\lambda(h_i)$ is a nonnegative integer for $1 \le i \le l$.

We denote the set of dominant integral linear functions by $\Lambda^+ = \{\lambda \in H^* \mid \lambda(h_i) \in \mathbb{N}_0 \text{ for } 1 \le i \le l\}$

Thm: If $\lambda \in \Lambda^+$, then the irreducible g-module $V(\lambda)$ is finite dimensional. Moreover, its set of weights $\Pi(\lambda)$ is permuted by $\mathcal{W}$ with $\dim V_\mu = \dim V_{\sigma\mu}$ for $\sigma \in \mathcal{W}$.

pf: <u>Step 1</u> $V(\lambda)$ contains a nonzero finite dimensional S_i-module for each $1 \le i \le l$.

$$\parallel$$
$$\langle v_\lambda, y_i v_\lambda, \cdots y_i^{\lambda(h_i)} v_\lambda \rangle$$

To see it is a S_i-module, we need to check it is stable under x_i, y_i, h_i.

- $x_i y_i^k v_\lambda = -k(k-1-\lambda(h_i))\, y_i^k v_\lambda$
- $y_i^{\lambda(h_i)+1} v_\lambda = 0$ Since $x_j y_i^{\lambda(h_i)+1} v_\lambda = 0$, $y_i^{\lambda(h_i)+1} v_\lambda$ is another maximal vector of $V(\lambda)$, contradicted with irreducibility.
- $h_i y_i^k v_\lambda = (-2k + \lambda(h_i)) y_i^{k+1} v_\lambda$

<u>Step 2</u> $V(\lambda)$ is the sum of all its finite dimensional S_i-submodules for each $1 \le i \le l$. Denoted by V'

It is non-zero by Step 1. Let W be any finite dimensional S_i-submodule of $V(\lambda)$.

Note that $x_\alpha W$ for $\alpha \in \phi$ is still finite dimensional, S_i-stable. Then $V(\lambda) = V'$ by irreducibility.

<u>Step 3</u>: $s_i = \exp\phi(x_i) \exp\phi(-y_i) \exp\phi(x_i)$ is a well-defined automorphism of $V(\lambda)$.

Indeed, for any $v \in V$, v lies in a finite sum of finite dimensional S_i-submodules, hence in a finite dimensional S_i-module. It implies $\phi(x_i)$ and $\phi(y_i)$ are locally nilpotent endomorphisms of V for $1 \le i \le l$.

Step 4: If u is any weight of V, then $S_i(V_u) = V_{\sigma_i u}$

Since V_u is finite dimensional, lies in a finite sum of finite dimensional S_i- submodules, hence in a finite dimensional S_i- module N. Then $S_i|_{V_u}$ sends to the weight vectors of weight $\sigma_i(u)$.

Then $S_i(V_u) = V_{\sigma_i u}$ by S_i is an automorphism.

Step 5: The set of weights $\Pi(\lambda)$ is finite. By Step 4, $\Pi(\lambda)$ is stable under $\mathcal{W}$, and $\dim V_u = \dim V_{\sigma u}$.

Consider $\Lambda = \{ \lambda \in H^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for } 1 \le i \le \ell \}$. $\forall u \in \Lambda$, is conjugate under $\mathcal{W}$ to one and only one dominate integeral weight. Moveover, $\forall u \in \Pi(\lambda)$, the dominant integral weight $\prec \lambda$. Using the fact: the set of $\mathcal{W}$- conjugates of all dominant integral linear function $u \prec \lambda$ is finite.

Conclusion: Since $\dim V_u$ is finite for all $u \in \Pi(\lambda)$, and $\Pi(\lambda)$ is finite, then $\dim V$ is finite.

Cor: The map $\lambda \mapsto V(\lambda)$ induces a one-one correspondence between Λ^+ and the isomorphism classes of finite dimensional irreducible $\mathfrak{g}$- modules.

Now we remain $V(\lambda)$ in the finite dimensional situation, i.e. $\lambda \in \Lambda^+$.

Prop: $u \in \Lambda$ belongs to $\Pi(\lambda) \Leftrightarrow u$ and all its $\mathcal{W}$- congugates be $\prec \lambda$.

pf: "$\Rightarrow$" obvious, since $\Pi(\lambda)$ is permuted by $\mathcal{W}$.

"$\Leftarrow$" $u \prec \lambda$, $\exists\, \sigma_0 \in W$ st. $\sigma_0 u$ is dominant integral $\prec \lambda$, then $\sigma_0 u \in \Pi(\lambda)$ Since $\lambda - \sigma_0 u = \sum_{i=1} \mathbb{Z}_+ \alpha_i$

Let $u \in \Pi(\lambda)$, by Weyl's complete reducibility theorm, u lies in some irreducible finite $\mathfrak{g}$-module.

Consider $V_{u+i\alpha}$ $(i \in \mathbb{Z})$ is invariant under S_i. the the weights in $\Pi(\lambda)$ of the form $u + i\alpha$ form a connected string. Moreover, the reflection σ_α reverses this string. Then $u = \sigma_0^{-1} \sigma_0 u \in W \pi(\lambda) = \Pi(\lambda)$

$\boxed{\text{Q: See an example?}}$

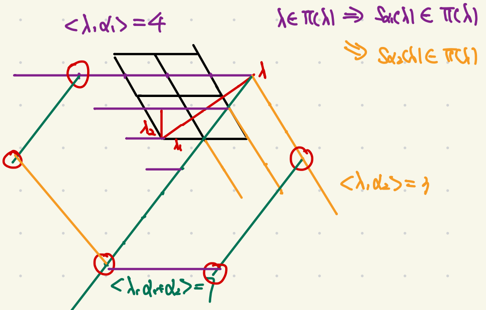
Consider $\mathfrak{g} = sl_3$. $W = \{ S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_1} S_{\alpha_2} \}$ $\lambda \in \Lambda^+$, can be written as a linear combinition of λ_1, λ_2

where $\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ $\qquad \lambda = 4\lambda_1 + 3\lambda_2$

$\lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$

$\alpha_1 = \frac{1}{3}(2\lambda_1 - \lambda_2)$

$\alpha_2 = \frac{1}{3}(-\lambda_1 + 2\lambda_2)$



Weight diagram

Q: How to characterise $V(\lambda)$ by generators and relations?

Recall that there is a canonical homomorphism of left $U(g)$-module $\varphi: U(g)/I(\lambda) \longrightarrow M(\lambda) = U(g) \otimes_{U(B)} V_\lambda$

$$\bar{1} \mapsto 1 \otimes v_\lambda$$

where $I(\lambda)$ is the left ideal of $U(g)$ generated by $x_\alpha (\alpha > 0)$ and $h_\alpha - \lambda(h_\alpha).1$ $(\alpha \in \Phi)$

By PBW basis, $\varphi(\bar{g}) = g \otimes v_\lambda = \underset{U(N^-)}{g_1} \underset{U(B)}{g_2} \otimes v_\lambda = g_1 \otimes g_2 v_\lambda = 0 \Rightarrow g \in U(B)$ or $g_2 v_\lambda = 0$ $\Rightarrow \bar{g} = 0$ i.e. $M(\lambda) \cong U(g)/I(\lambda)$

Equivalently, $I(\lambda)$ is the annihilator of a maximal vector v_λ of weight λ.

Now fix a $\lambda \in \Lambda^+$, let $J(\lambda)$ be the left ideal in $U(g)$ which annihilates a maximal vector of weight λ.

The inclusion $I(\lambda) \subset J(\lambda)$ induces the canonical map $Z(\lambda) = U(g)/I(\lambda) \longrightarrow U(g)/J(\lambda) \cong V(\lambda)$

Thm: Let $\lambda \in \Lambda^+$, then $J(\lambda)$ is generated by $I(\lambda)$ along with all $y_i^{\lambda(h_i)+1}$.

pf: Let $J'(\lambda)$ is generated by $I(\lambda)$ along with all $y_i^{\lambda(h_i)+1}$.

<u>Step 1</u>: $V'(\lambda) = U(g)/J'(\lambda)$ is finite dimensional. Thanks to the proof of the above Thm, it suffice to show $V'(\lambda)$ is a sum of finite dimensional S_i-submodules. It is equivalent to show each y_i and x_i is locally nilpotent on $V'(\lambda)$. For x_i it is obvious, since we cannot have $\mu + k\alpha_i < \lambda$ for all $k \geq 0$.

For y_i, by PBW thm, $V'(\lambda)$ is spanned by the cosets of $y_{i_1} \cdots y_{i_t}$. $(1 \leq i_j \leq \ell)$

Induction on length of monomials, starting at 1, then proves the local nilpotence of y_i.

Claim that if $y_{i_1} \cdots y_{i_t}$ is killed by y_i^k, then the longer monomial $y_{i_0} \cdots y_{i_t}$ is killed by y_i^{k+3}.

Indeed, $y_i^{k+3} y_{i_0} \cdots y_{i_t} \bar{1} = [y_i^{k+3}, y_{i_0}] y_{i_1} \cdots y_{i_t} \bar{1} + y_{i_0} y_i^{k+3} y_{i_1} \cdots y_{i_t} \bar{1} = 0$

$$[y_i^{k+3}, y_{i_0}] = \left((k+2)[y_i, y_{i_0}] y_i^{k+2} + \binom{k+3}{2}[y_i, [y_i, y_{i_0}]] y_i^{k+1} + \binom{k+3}{3}[y_i, [y_i, [y_i, y_{i_0}]]] y_i^k\right) y_{i_1} \cdots y_{i_t} \bar{1}$$

Since root strings have length at most 4.

<u>Step 2</u>: On one hand, $V'(\lambda)$ is standard cyclic, so it is indecomposable.

On the other hand, $V(\lambda)$ is finite dimensional, so it is complete reducible.

Hence, $V'(\lambda)$ is irreducible. $J(\lambda) \subset J'(\lambda)$ implies that $V(\lambda)$ is a homomorphic image of $V'(\lambda)$

It forces $V(\lambda) = V'(\lambda)$ by $V(\lambda)$ is irreducible.

Q: As is usually done, what's the character of $V(\lambda)$?

Recall that the formal character $ch_{V(\lambda)}$ is defined by $\sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) e^\mu$, where $m_\lambda(\mu) = \dim V(\lambda)_\mu$.

e^μ is the basis of group ring $\mathbb{Z}(\Lambda)$ with addition

multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$ and extend by linearity.

W acts naturally on $\mathbb{Z}(\Lambda)$ by $\sigma e^\mu = e^{\sigma\mu}$

The character of $M(\lambda)$ is easy to see, since the weight of basis vectors are $\lambda - \sum_{\alpha \in \Phi^+} k_\alpha \alpha$

Then $ch_{M(\lambda)} = e^{\lambda}\prod_{\alpha\in\Phi^+} e^{-k_\alpha\alpha} = e^{\lambda}\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{-1}$. We denote $\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})$ by Δ.

Thm: (Weyl Character Formula)

$$ch_{V(\lambda)} = e^{\rho}\Delta^{-1}\sum_{w\in W} \operatorname{sgn}(w)\, e^{w(\lambda+\rho)}$$

pf Step 1: Considering the composition series of $M(\lambda)$, then $ch_{M(\lambda)} = \sum_{\mu\in h^*}[M(\lambda), V(\mu)]\, ch_{V(\mu)}$, where $[M(\lambda), V(\mu)]$ is the number of times $V(\mu)$ appear as a composition factor. Note that if $[M(\lambda), V(\mu)]\neq 0$, we must have $\mu\preceq\lambda$. Since each $V(\mu)$ as a subquotient of $L(\lambda)$, and Casimir element Ω acts on $M(\lambda)$ by scalar $\langle\lambda+\rho, \lambda+\rho\rangle - \langle\rho,\rho\rangle$. Then we have $\langle\lambda+\rho, \lambda+\rho\rangle = \langle\mu+\rho, \mu+\rho\rangle$ if $[M(\lambda), L(\mu)]\neq 0$. Let $S = \{\mu\in h^* \mid \mu\preceq\lambda, \langle\mu+\rho, \mu+\rho\rangle = \langle\lambda+\rho, \lambda+\rho\rangle\}$, we can rewrite by $ch_{M(\lambda)} = \sum_{\mu\in S} a_{\lambda\mu}\, ch_{V(\mu)}$, where $a_{\lambda\mu} = [M(\lambda): V(\mu)]$, order by total ordering inherited from the partial order $\preceq$.

Step 2: Consider the infinite matrix $(a_{\lambda\mu})$ has nonnegative integer entries, is upper triangular with $a_{\lambda\lambda}=1$ along the diagonal. Since $(a_{\lambda\mu})$ is invertible, then $ch_{V(\lambda)} = \sum_{\mu\in S} b_{\lambda\mu}\, ch_{M(\mu)}$. for some integer coefficents $b_{\lambda\mu}$. Substitude $ch_{M(\mu)}$ by $e^{\mu}\Delta^{-1}$, we have that $e^{\rho}\Delta\, ch_{V(\lambda)} = \sum_{\mu\in S} b_{\lambda\mu}e^{\mu+\rho}$. Take $s_i\in W$, $s_i(e^{\rho}\Delta) = e^{\rho-\alpha_i}(1-e^{\alpha_i})\prod_{\alpha\in\Phi^+\setminus\{\alpha_i\}}(1-e^{\alpha_i}) = e^{\rho}(e^{-\alpha_i}-1)\prod_{\alpha\in\Phi^+\setminus\{\alpha_i\}}(1-e^{-\alpha}) = -e^{\rho}\Delta$. For any $w\in W$, $w(e^{\rho}\Delta) = \operatorname{sgn}(w)e^{\rho}\Delta$.

Step 3: On one hand, $w(\sum_{\mu\in S} b_{\lambda\mu}e^{\mu+\rho}) = \sum_{\mu\in S} b_{\lambda\mu}e^{w(\mu+\rho)}$. On the other hand, $w(\sum_{\mu\in S} b_{\lambda\mu}e^{\mu+\rho}) = \operatorname{sgn}(w)\sum b_{\lambda\mu}e^{\mu+\rho}$. Hence $b_{\lambda\mu} = \operatorname{sgn}(w)\, b_{\lambda\nu}$ if $w(\mu+\rho) = \nu+\rho$ for some $w\in W$. Let $T_\mu = \{\nu\in h^* \mid w(\mu+\rho) = \nu+\rho$ for some $w\in W\}$. Pick a weight ν such that the height of $\lambda-\nu$ is minimal. Claim that $\nu+\rho$ be a dominant integral weight. Moreover, $\nu=\lambda$. Indeed, If $\lambda,\mu\in\Lambda^+$ such that $\lambda(h_i)>0$ for $1\le i\le \ell$, $\mu\preceq\lambda$, and $\langle\lambda,\lambda\rangle = \langle\mu,\mu\rangle$, then $\lambda=\mu$. Consider $\nu\in S$, then $\langle\nu+\rho, \nu+\rho\rangle = \langle\lambda+\rho, \lambda+\rho\rangle$, and $(\lambda+\rho)(h_i)>0$ for $1\le i\le\ell$, $\nu+\rho\preceq\lambda+\rho$. It implies $\nu=\lambda$.

Step 4: Until now, we know that for any weight such that $b_{\lambda\mu}\neq 0$, there is $w\in W$ such that $\mu+\rho = w(\lambda+\rho)$. Then $b_{\lambda\mu} = \operatorname{sgn}(w)\, b_{\lambda\lambda} = \operatorname{sgn}(w)$. It implies $\sum_{\mu\in S} b_{\lambda\mu}e^{\mu+\rho} = \sum_{w\in W}\operatorname{sgn}(w)\, e^{w(\lambda+\rho)}$, we have done ^^.